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A NEUTRAL TWO-FLAVOR LOFF COLOR SUPERCONDUCTOR

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Abstract

In this paper we construct analytically a LOFF color superconducting state that is both color and charge neutral using the weak coupling approximation. We demonstrate that this state is free from chromomagnetic instabilities. Its relevance to the realistic quark matter at moderately high baryon density is discussed.

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The ground state of quark matter at moderately high baryon density and very low temperature has been an active area of research recently because of its relevance to the interior of a compact star. While a color superconductor [1] is expected, it is unlikely to be of the conventional BCS type because the pairing occurs between quarks of different flavors and Fermi momenta. The physics of Cooper pairing among unequal Fermi momenta applies also in systems of cold atomic gasses [2] as well. Several exotic color superconducting states have been proposed as candidate ground states. These include the gapless CSC state [3][4][5], the LOFF-CSC state [6], and a state that consists of a heterogeneous mixture of gapped CSC and normal phases [7]. But none is completely satisfactory for different reasons. The gapless CSC state-a generalization of the Sarma state in solid state physics-although being stable against small variations of the gap parameters and with the condition of local charge neutrality fully implemented, it suffers from chromomagnetic instabilities when coupled to the gluon-photon field [8] [9][10][11][12][13]. In contrast, the LOFF state, although is free from chromagnetic instabilities [12] the charge neutrality condition has to be implemented and its free energy may favor a complicated superposition of plane waves, which makes even numerical treatment difficult [14]. The mixed phase scenario, being an alternative way to avoid the problem of chromomagnetic instabilities, has failed to make quantitative comparisons between its free energy with its homogeneous competitors.

In a previous paper[12], we demonstrated the chromomagnetic stability of a simple two-flavor LOFF state consisting of a single plane wave, within the region of the displacement parameter where it is energetically favored compared to the BCS and normal phases. The analysis was carried out using a grand canonical ensemble at fixed baryon and electric chemical potentials while the 8th-color chemical potential was set to zero. In this paper, we shall construct analytically a two flavor LOFF state using the canonical ensemble at fixed baryon density and we shall implement the color and charge neutrality conditions. In order to justify the weak coupling approximation that we shall employ, we shall treat the sum of the electric charges of the u and d quarks as free parameter ϵ, more specifically we write

$$Q_u = \frac{1}{3} + \epsilon \qquad Q_d = -\frac{1}{3}. \tag{1}$$

The accuracy of the weak coupling approximation requires that $\epsilon << 1$, but we expect that the qualitative conclusions as to the existence of a neutral and stable LOFF state drawn in this paper can be extrapolated to the realistic case $\epsilon = \frac{1}{3}$ of the real world.

The dynamics of quark matter with two flavors can be described by the following NJL effective Lagrangian

$$\begin{aligned}\mathcal{L} = -\bar{\psi}\gamma_\mu\frac{\partial\psi}{\partial x_\mu} + \bar{\psi}\gamma_4\mu\psi \quad &+ \quad G_S[(\bar{\psi}\psi)^2 + (\bar{\psi}\vec{\tau}\psi)^2] \\ &+ \quad G_D(\bar{\psi}_C\gamma_5\epsilon^c\tau_2\psi)(\bar{\psi}\gamma_5\epsilon^c\tau_2\psi_C),\end{aligned} \tag{2}$$

where ψ represents the quark fields and $\psi_C = C\tilde{\bar{\psi}}$ its charge conjugate with $C = i\gamma_2\gamma_4$. All gamma matrices are hermitian, $(\epsilon^c)^{mn} = \epsilon^{cmn}$ is a 3×3 matrix acting on the red(r), green(g) and blue(b) color indices and the Pauli matrices $\vec{\tau}$ act on the u and d flavor (isospin) indices.

The chemical potential written in a matrix form in the color-flavor space reads

$$\mu = \frac{1}{3}\mu_B + \mu_Q Q + \mu_8 \lambda_8, \tag{3}$$

or in component form

$$\begin{aligned} \mu_u^{r,g} &= \frac{1}{3}\mu_B + \Big(\frac{1}{3}+\epsilon\Big)\mu_Q + \frac{1}{2\sqrt{3}}\mu_8, \\ \mu_d^{r,g} &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q + \frac{1}{2\sqrt{3}}\mu_8, \\ \mu_u^{b} &= \frac{1}{3}\mu_B + \Big(\frac{1}{3}+\epsilon\Big)\mu_Q - \frac{1}{\sqrt{3}}\mu_8, \\ \mu_d^{b} &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \frac{1}{\sqrt{3}}\mu_8, \end{aligned} \tag{4}$$

where μ_B denotes the chemical potential associated with the baryon number, μ_Q with the electric charge and μ_8 with the eighth color. In terms of the notation of Ref. [12], i.e. $\mu = \bar{\mu} - \delta\tau_3 + \delta'\lambda_8$. We have

$$\begin{aligned} \bar{\mu} &= \frac{1}{3}\mu_B + \frac{1}{2}\epsilon\mu_Q, \\ \delta &= -\Big(\frac{1}{3}+\frac{1}{2}\epsilon\Big)\mu_Q \simeq -\frac{1}{3}\mu_Q, \\ \delta' &= \mu_8, \end{aligned} \tag{5}$$

where $\bar{\mu} = \frac{1}{6}\mathrm{tr}\mu$ denotes the mean chemical potential of all colors and flavors, δ measures the displacement between the Fermi momenta of the two flavors and δ' is proportional to the displacement between the blue color and the other colors.

In the presence of a diquark condensate,

$$< \bar{\psi}(\vec{r})\gamma_5\lambda_2\tau_2\psi_C(\vec{r}) >= \Phi e^{2i\vec{q}\cdot\vec{r}}, \tag{6}$$

the pressure at $T=0$ can be calculated using the mean field approximation. We find that

$$p = p_n + p_{\text{cond.}} \tag{7}$$

where p_n refers to the contribution of the normal phase and is approximated by an ideal gas of quarks, i.e.

$$\begin{aligned} p_n &= \frac{1}{12\pi^2}\mathrm{tr}\mu^4 = \\ \tfrac{1}{12\pi^2} & \Big[2\Big(\bar{\mu}-\delta+\frac{\delta'}{2\sqrt{3}}\Big)^4 + 2\Big(\bar{\mu}+\delta+\frac{\delta'}{2\sqrt{3}}\Big)^4 + \Big(\bar{\mu}-\delta-\frac{\delta'}{\sqrt{3}}\Big)^4 + \Big(\bar{\mu}+\delta-\frac{\delta'}{\sqrt{3}}\Big)^4\Big]. \end{aligned} \tag{8}$$

The contribution from the condensate, $p_{\text{cond.}}$ was computed in Ref. [12] for $\delta'=0$ in the weak coupling approximation

$$\begin{aligned} p_{\text{cond.}} &= -\Gamma(\bar{\mu},\Delta,\delta,q) \\ &\simeq -\frac{2\bar{\mu}^2}{\pi^2}\Big\{\Delta^2\Big(\ln\frac{\Delta}{\Delta_0}-\frac{1}{2}\Big) + \frac{(q+\delta)^3}{4q}\Big[(1-x_1^2)\ln\frac{1+x_1}{1-x_1} + \frac{2}{3}(2x_1^3-3x_1+1)\Big] \\ &+ \frac{(q-\delta)^3}{4q}\Big[(1-x_2^2)\ln\frac{1+x_2}{1-x_2} + \frac{2}{3}(2x_2^3-3x_2+1)\Big]\Big\}, \end{aligned} \tag{9}$$

where x_1 and x_2 are dimensionless parameters that were introduced in Ref. [15], i.e.

$$x_1 = \theta\Big(1 - \frac{\Delta}{(q+\delta)}\Big)\sqrt{1 - \frac{\Delta^2}{(q+\delta)^2}} \tag{10}$$

$$x_2 = \theta\Big(1 - \frac{\Delta}{|q-\delta|}\Big)\sqrt{1 - \frac{\Delta^2}{(q-\delta)^2}}. \tag{11}$$

and Δ_0 is the BCS gap energy given by the expression

$$\frac{1}{4G_D} = \frac{2\bar{\mu}^2}{\pi^2}\ln\frac{2\omega_0}{\Delta_0} \tag{12}$$

where ω_0 represents the UV cutoff of the pairing force. In a grand canonical ensemble approach, the pressure ought to be maximized with respect to Δ and q at equilibrium. The weak coupling approximation is justified if the conditions that $\Delta_0 << \mu_0$ and Δ, δ, and q are comparable or smaller than Δ_0 are satisfied. Since we shall employ the weak coupling approximation, only terms up to the order of $\bar{\mu}^2\Delta^2$ are retained in the expression for the pressure.

The equilibrium conditions

$$\Big(\frac{\partial p}{\partial \Delta^2}\Big)_{\bar{\mu},\delta,\delta',q} = 0, \qquad \Big(\frac{\partial p}{\partial q}\Big)_{\bar{\mu},\delta,\delta',\Delta} = 0, \tag{13}$$

allow three types of CSC solutions:

1) The 2SC state (BCS state). This solution is characterized by $q = 0$ and $\Delta = \Delta_0$. The contribution of the condensate to the pressure is given by

$$p_{\text{cond.}} = \frac{\bar{\mu}^2}{\pi^2}(\Delta_0^2 - 2\delta^2). \tag{14}$$

This solution corresponds to the range of the displacement parameter $\delta < \Delta_0$.

2) The g2SC state, which is characterized by $q = 0$, and $\Delta = \sqrt{\Delta_0(2\delta - \Delta_0)}$. The expression for the pressure is

$$p_{\text{cond.}} = -\frac{\bar{\mu}^2}{\pi^2}(2\delta - \Delta_0)^2. \tag{15}$$

It corresponds to the range $\Delta_0/2 < \delta < \Delta_0$ for the displacement parameter.

3) The 2SC-LOFF state which is characterized by nonzero values for q and Δ. Within the LOFF window, $\delta_c' < \delta < \delta_c$ with [15]

$$\delta_c' \simeq 0.706\Delta_0 \qquad \delta_c \simeq 0.754\Delta_0. \tag{16}$$

The 2SC-LOFF state is both energetically favored and free from chromomagnetic instabilities [12].

The charge neutrality condition,

$$\Big(\frac{\partial p}{\partial \mu_Q}\Big)_{\mu_B,\mu_8} = \frac{1}{2}\epsilon\Big(\frac{\partial p}{\partial \bar{\mu}}\Big)_{\delta,\delta'} - \Big(\frac{1}{3} + \frac{1}{2}\epsilon\Big)\Big(\frac{\partial p}{\partial \delta}\Big)_{\bar{\mu},\delta'} = 0 \tag{17}$$

gives rise to different values of the displacement parameter δ for the normal and different CSC phases for a given value of μ_B. In the interesting region where $\delta \sim \Delta_0$, the impact of the different δ's on the pressure cannot be ignored within the weak coupling approximation and the phase balance has to be reconsidered. This task will be carried out below. For the sake of analytical tractability, we shall restrict our attention mainly to the values of δ of the 2SC-LOFF phase slightly below the threshold δ_c. But we leave aside the possibility of a LOFF state with muti-plane waves. The pressure from the LOFF condensate in this case reads

$$p_{\rm cond.} = \frac{4\bar{\mu}^2}{\pi^2}(\rho_c^2 - 1)(\delta - \delta_c)^2, \tag{18}$$

with $\frac{q}{\delta} = \rho_c \simeq 1.20$ given by the equation

$$\frac{1}{\rho_c}\ln\frac{\rho_c + 1}{\rho_c - 1} = 2. \tag{19}$$

We shall continue to work with the grand canonical ensemble for fixed μ_B and we shall retain the approximation $\delta' = 0$. The difference between this approach and the canonical ensemble with fixed baryon number density as well as the correction that a nonzero δ' induces, dictated by the color neutrality are beyond the weak coupling approximation as we shall argue subsequently.

In what follows, the pressure of a charge neutral state will be denoted by a capital letter P.

The pressure of the normal phase is approximated by (8) at $\delta' = 0$, i.e.

$$p_n \simeq \frac{\bar{\mu}^2}{2\pi^2}(\bar{\mu}^2 + 6\delta^2), \tag{20}$$

where we have dropped higher order terms in δ. The charge neutrality condition (17) yields

$$\delta = \frac{1}{2}\epsilon\bar{\mu} \equiv \delta_n \tag{21}$$

and the pressure of the charge neutral normal phase becomes

$$P_n = \frac{\bar{\mu}^4}{2\pi^2}(1 + \frac{3}{2}\epsilon^2). \tag{22}$$

The pressure of the LOFF state is the sum of eq.(20) and eq. (18) for δ close to the upper edge of the LOFF window. Consequently we can write

$$p_{\rm LOFF} = p_n + \frac{4\bar{\mu}^2}{\pi^2}(\rho_c^2 - 1)(\delta_c - \delta)^2. \tag{23}$$

The solution to the charge neutral condition (17) is

$$\delta = \frac{3\delta_n + 4(\rho_c^2 - 1)\delta_c}{4\rho_c^2 - 1} \equiv \delta_{\rm LOFF}. \tag{24}$$

and we have

$$P_{\rm LOFF} = P_n + \frac{9}{2}\Big(\frac{\partial^2 p_n}{\partial\mu_Q^2}\Big)_{\mu_B,\mu_8}|_{\mu_Q=-3\delta_n}(\delta_{\rm LOFF}-\delta_n)^2 + \frac{4\bar\mu^2}{\pi^2}(\rho_c^2-1)(\delta_c-\delta_{\rm LOFF})^2$$
$$= P_n + \frac{12\bar\mu^2}{\pi^2}\frac{\rho_c^2-1}{4\rho_c^2-1}(\delta_c-\delta_n)^2. \tag{25}$$

Note that the contribution of the difference $\delta_{\rm LOFF}-\delta_n$ to the pressure is of the same order as $p_{\rm cond.}$. It follows from (16) and (21) that

$$\epsilon\bar\mu \simeq 1.508\Delta_0 \tag{26}$$

for $\delta_n \simeq \delta_c$.

The pressure of the 2SC phase is given by the sum of (20) and eq.(14), i.e.

$$p_{\rm 2SC} \simeq \frac{\bar\mu^2}{2\pi^2}(\bar\mu^2 + 2\delta^2 + 2\Delta_0^2). \tag{27}$$

It follows from (17) that

$$\delta = \frac{3}{2}\epsilon\bar\mu \equiv \delta_{\rm 2SC}. \tag{28}$$

Comparing this state with the neutral 2SC-LOFF state by combining (26) and (28) we observe that $\delta_{\rm 2SC} = 2.26\Delta_0$, which lies outside the range of the 2SC solution. Stated differently, the 2SC state does not exist for the parametric relation (26).

The pressure of the g2SC phase is given by the sum of (20) and (15). We find

$$p_{\rm g2SC} \simeq \frac{\bar\mu^2}{2\pi^2}(\bar\mu^2 + 8\Delta_0\delta - 2\delta^2 - 2\Delta_0^2]. \tag{29}$$

The charge neutrality (17) yields

$$\delta = -\frac{3}{2}\epsilon\bar\mu + 2\Delta_0 \equiv \delta_{\rm g2SC}. \tag{30}$$

Again comparing it with the neutral 2SC-LOFF at the same μ_B, we find that $\delta_{\rm g2SC} \simeq -0.26\Delta_0$ and the g2SC solution is similarly ruled out.

Therefore we have established that the neutral LOFF state is both energetically favorable and chromomagnetically stable for a given baryonic chemical potential μ_B at $\mu_8 = 0$.

In a canonical ensemble approach of the quark matter at a fixed baryon density, n_B, the thermodynamic quantity which is minimized is the density of the Helmholtz free energy

$$f = \mu_B n_B - p \tag{31}$$

subject to the condition that

$$n_B = \Big(\frac{\partial p}{\partial\mu_B}\Big)_{\mu_Q,\mu_8} = \frac{1}{3}\Big(\frac{\partial p}{\partial\bar\mu}\Big)_{\delta,\delta'}, \tag{32}$$

which generates slightly different μ_B for different phases. It follows from (9) that the shift of μ_B in a super phase from that of the normal phase, i.e.

$$\delta\mu_B \equiv (\mu_B)_s - (\mu_B)_n \sim \Delta^2/(\bar\mu)_n \tag{33}$$

with

$$(\bar\mu)_n \simeq \Big(\frac{3\pi^2 n}{2}\Big)^{\frac{1}{3}}. \tag{34}$$

The term of order $\delta\mu_B$ is cancelled by the Legendre transformation on the RHS of (31). The leading contribution of $\delta\mu_B$ to the Helmholtz free energy is of order $\mu_B^2\delta\mu_B^2 \sim \Delta^4$ for all CSC states and therefore can be neglected.

The color neutrality condition

$$\Big(\frac{\partial p}{\partial \delta'}\Big)_{\bar\mu,\Delta,\delta} = 0 \tag{35}$$

yields $\delta' = \mu_8 = 0$ in the normal phase following the expression (8). To asses its impact in the super phase, we have to generalize our previous result, eq. (9) for a nonzero δ'. Since only red and green quarks participate in the pairing and a nonzero δ' shifts their mean chemical potential without offseting the Fermi momentum displacement, we have

$$p_{\rm cond.} = -\Gamma\Big(\bar\mu + \frac{\delta'}{2\sqrt{3}}, \Delta, \delta, q\Big) \tag{36}$$

A small value of

$$\delta' = \delta'_s \sim \frac{\Delta^2}{\bar\mu} \tag{37}$$

is induced in the super phase according to equ. (35) and contributes a term of order $\Big(\frac{\partial^2 p_n}{\partial\delta'^2}\Big)|_{\delta'=0}\delta_s'^2 \sim \Delta^4$ to the pressure or the free energy of all CSC states, which is agian negligible.

A sublety arises regarding the Meissner tensors of the 4-7th gluons. In the previous paper [12], we derived the small Δ/δ expansion of the Meissner mass tensors at $\delta' = 0$. We found that

$$(m_4^2)_{ij} = (m_5^2)_{ij} = (m_6^2)_{ij} = (m_7^2)_{ij} = A(\delta_{ij} - \hat q_i\hat q_j) + B\hat q_i\hat q_j \tag{38}$$

and

$$(m_8^2)_{ij} = C(\delta_{ij} - \hat q_i\hat q_j) + D\hat q_i\hat q_j, \tag{39}$$

where

$$A = \frac{g^2\bar\mu^2}{96\pi^2}\frac{1}{(\rho_c^2-1)^2}\Big(\frac{\Delta}{\delta}\Big)^4 \ge 0, \qquad B = \frac{g^2\bar\mu^2}{8\pi^2}\frac{1}{\rho_c^2-1}\Big(\frac{\Delta}{\delta}\Big)^2 \ge 0 \tag{40}$$

and

$$C = 0, \qquad D = \frac{g^2\bar\mu^2}{6\pi^2}\Big(1+\frac{e^2}{3g^2}\Big)\frac{1}{\rho_c^2-1}\Big(\frac{\Delta}{\delta}\Big)^2 \ge 0 \tag{41}$$

at equilibrium. The absence of a Δ^2 term in A follows from the relation between the one loop diagram for the quadratic term of A and that of the quadratic term of C through an integration by parts when $\delta' = 0$. The latter vanishes at equilibrium because of the

second condition of eqs.(13). This relation breaks down for a nonzero δ' since the blue quarks enter the one loop diagram of A but not that of C. It can be shown explicitly that A acquires a Δ^2 term of order $\bar{\mu}^2\Delta^2\delta'^2/\delta^4$, which is suppressed relative to the expression of A of eq.(40) at weak coupling following the estimation (37). Therefore the property that the neutral 2SC-LOFF state is free from chromomagnetic instabilities remains intact under the color neutrality constraint.

In conclusion, we have found analytically a 2SC-LOFF state that is both color and electric neutral and free from chromomagnetic instabilities. The controlling small parameter of the weak coupling approximation employed is the charge sum of the u and d flavors, which is of order Δ_0/μ_B for the above mentioned LOFF state. For realistic quark matter where the charge sum is $\frac{1}{3}$, we have

$$\frac{\Delta_0}{\bar{\mu}} \simeq 0.22 \tag{42}$$

in accordance with (26). While the weak coupling approximation becomes marginal, it is still instructive to compare our result with the numerical analysis performed by Shovkovy and Huang [3]. It appears that our case corresponds to the weaker NJL coupling side of their results, where the line of charge neutrality on the $\Delta - \delta$ plane excludes both the 2CS and g2SC states. We find that a neutral LOFF state survives there. There are two possible future directions to follow. The first one is within the framework of the weak coupling approximation. One might attempt to extend the analysis to the lower value of the displacement parameter δ and explore the phase balance of the 2SC-LOFF and other exotic CSC states. In particular, because of different phases have different values of δ, the LOFF window specified by (16) will be modified. It would be very interesting if the window could become wider. Alternatively, one may relax the week coupling approximation and perform a numerical analysis including the 2SC-LOFF state similar to that of Ref. [3]. In this manner more quantitative results can be obtained for realistic quark matter.

After this work was completed, reference [16] appeared where a neutral LOFF state of three flavors is discussed.

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